https://www.linkedin.com/feed/update/urn:li:activity:7048223280259829760?utm_source=share&utm_medium=member_desktop
Evaluate

$$L = \lim_{n\to\infty} \frac{\sqrt[n(n+1)]{1!2!3!\dots n!}}{\sqrt{n}}.$$

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Denoting $sf(n) := 1!2!3!\dots n!$ (Superfactorial), $H(n) := 1^1 2^2 \dots n^n$ (Hyperfactorial)
we obtain $sf(n) = 1^n 2^{n-1} 3^{n-2} \dots (n-1)^2 n^1 = \dfrac{1^{n+1} 2^{n+1} 3^{n+1} \dots (n-1)^{n+1} n^{n+1}}{1^1 2^2 3^3 \dots (n-1)^{n-1} n^n} = \dfrac{(n!)^{n+1}}{H(n)}$.

Lemma.
$\sqrt[n]{H(n)} \sim e^{-\frac{n}{4}} n^{\frac{n+1}{2}}$ ("$\sim$" asymptotically equivalent), that is $\lim_{n\to\infty}\left(e^{\frac{n}{4}} n^{-\frac{n+1}{2}} \sqrt[n]{H(n)}\right) = 1$.

Proof.
Let $b_n := e^{\frac{n^2}{4}} n^{-\frac{n(n+1)}{2}} H(n)$. By **Multiplicative Stolz Theorem** if sequence $\left(\dfrac{b_{n+1}}{b_n}\right)_{n\geq 1}$ convergence then $\lim_{n\to\infty}\left(e^{\frac{n}{4}} n^{-\frac{n+1}{2}} \sqrt[n]{H(n)}\right) = \lim_{n\to\infty} \sqrt[n]{b_n} = \lim_{n\to\infty} \dfrac{b_{n+1}}{b_n}$.

We have $\dfrac{b_{n+1}}{b_n} = \dfrac{e^{\frac{(n+1)^2}{4}} (n+1)^{-\frac{(n+1)(n+2)}{2}} H(n+1)}{e^{\frac{n^2}{4}} n^{-\frac{n(n+1)}{2}} \cdot H(n)} = \dfrac{e^{\frac{2n+1}{4}} n^{\frac{n(n+1)}{2}} (n+1)^{n+1}}{(n+1)^{\frac{(n+1)(n+2)}{2}}} =$
$\dfrac{e^{\frac{2n+1}{4}} n^{\frac{n(n+1)}{2}}}{(n+1)^{\frac{(n+1)(n+2)}{2}-(n+1)}} = \dfrac{e^{\frac{2n+1}{4}} n^{\frac{n(n+1)}{2}}}{(n+1)^{\frac{n(n+1)}{2}}} = \dfrac{e^{\frac{2n+1}{4}}}{\left(1+\frac{1}{n}\right)^{\frac{n(n+1)}{2}}}$ and then

$$\ln \frac{b_{n+1}}{b_n} = \frac{2n+1}{4} - \frac{n(n+1)}{2} \ln\left(1+\frac{1}{n}\right).$$

Since $\ln(1+t) = t - \dfrac{t^2}{2} + o(t^2)$ then $\ln \dfrac{b_{n+1}}{b_n} = \dfrac{2n+1}{4} - \dfrac{n(n+1)}{2}\left(\dfrac{1}{n} - \dfrac{1}{2n^2} + o\left(\dfrac{1}{n^2}\right)\right) =$
$\dfrac{2n+1}{4} - \dfrac{n+1}{2} + \dfrac{n+1}{4n} + \dfrac{n(n+1)}{2} \cdot o\left(\dfrac{1}{n^2}\right) = \dfrac{1}{4n} + \dfrac{n(n+1)}{2} \cdot o\left(\dfrac{1}{n^2}\right)$ and, therefore,
$\lim_{n\to\infty} \ln \dfrac{b_{n+1}}{b_n} = \lim_{n\to\infty}\left(\dfrac{1}{4n} + \dfrac{n(n+1)}{2} \cdot o\left(\dfrac{1}{n^2}\right)\right) = 0.$

Hence, $\lim_{n\to\infty}\left(e^{\frac{n}{4}} n^{-\frac{n+1}{2}} \sqrt[n]{H(n)}\right) = e^0 = 1 \Leftrightarrow \lim_{n\to\infty}\left(\dfrac{n^{\frac{n+1}{2}}}{e^{\frac{n}{4}} \sqrt[n]{H(n)}}\right) = 1.$

As corollary from the **Lemma** we obtain

(**1**) $$\lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt[n(n+1)]{H(n)}} = \sqrt[4]{e}.$$

Indeed, $\lim_{n\to\infty}\left(\dfrac{n^{\frac{n+1}{2}}}{e^{\frac{n}{4}} \sqrt[n]{H(n)}}\right) = 1$ implies $\lim_{n\to\infty}\left(\dfrac{n^{\frac{n+1}{2}}}{e^{\frac{n}{4}} \sqrt[n]{H(n)}}\right)^{\frac{1}{n+1}} =$
$\lim_{n\to\infty} \dfrac{\sqrt{n}}{e^{\frac{n}{4(n+1)}} \sqrt[n(n+1)]{H(n)}} = 1 \Leftrightarrow \lim_{n\to\infty} \dfrac{\sqrt{n}}{\sqrt[n(n+1)]{H(n)}} \cdot \lim_{n\to\infty} \dfrac{1}{e^{\frac{n}{4(n+1)}}} = 1 \Leftrightarrow \lim_{n\to\infty} \dfrac{\sqrt{n}}{\sqrt[n(n+1)]{H(n)}} = e^{1/4}$.

Using (**1**) and taking in account that $\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n} = e^{-1}$ we finally obtain

$$L = \lim_{n\to\infty}\frac{\sqrt[n(n+1)]{sf(n)}}{\sqrt{n}} = \lim_{n\to\infty}\left(\frac{1}{\sqrt{n}} \cdot \sqrt[n(n+1)]{\frac{(n!)^{n+1}}{H(n)}}\right) = \lim_{n\to\infty}\left(\frac{\sqrt[n]{n!}}{n} \cdot \frac{\sqrt{n}}{\sqrt[n(n+1)]{H(n)}}\right) =$$

$$e^{-1} \cdot e^{1/4} = e^{-3/4}.$$